Sums with convolution of Dirichlet characters

Dmitry Ushanov

1 Introduction

Let χ_1 and χ_2 be two primitive Dirichlet characters with conductors q_1 and q_2 respectively. In a recent paper [1] Banks and Shparlinski considered the sum $S_{\chi_1,\chi_2}(T) = \sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y)$. For this sum they established upper bound

$$S_{\chi_1,\chi_2}(T) \ll T^{13/18} q_1^{2/27} q_2^{1/9+o(1)}$$

for $T \geqslant q_2^{2/3} \geqslant q_1^{2/3}$, and

$$S_{\chi_1,\chi_2}(T) \ll T^{5/8} q_1^{3/32} q_2^{3/16+o(1)}$$

for $T \geqslant q_2^{3/4} \geqslant q_1^{3/4}$.

In this paper we prove more precise bounds on S_{χ_1,χ_2}.

2 Statement of results

Theorem 1. *Let χ_1 and χ_2 be two primitive Dirichlet characters with conductors q_1 and q_2, respectively. If $q_1 \leqslant q_2$ and $T > 1$ then for every $\epsilon > 0$ one has*

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll \begin{cases} T^{2/3}(q_1q_2)^{1/9+\epsilon} & \text{if } (q_1q_2)^{1/3} \leqslant T \leqslant q_1^{4/3}q_2^{1/3}, \\ T^{3/4}q_2^{1/12+\epsilon} & \text{if } q_1^{4/3}q_2^{1/3} \leqslant T, \end{cases} \quad (1)$$

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll \begin{cases} T^{1/2}(q_1q_2)^{3/16+\epsilon} & (q_1q_2)^{3/8} \leqslant T \leqslant q_1^{9/8}q_2^{3/8}, \\ T^{2/3}q_2^{1/8+\epsilon} & q_1^{9/8}q_2^{3/8} \leqslant T. \end{cases} \quad (2)$$

(Constant implied by $\ll$ depends only on ϵ)

Collorally 1. *Suppose that under the conditions of Theorem 1 we have $q_1 = q_2 = q$. Then*

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll \begin{cases} T^{2/3}q^{2/9+\epsilon} & \text{if } q^{2/3} \leqslant T \leqslant q^{11/12}, \\ T^{1/2}q^{3/8+\epsilon} & \text{if } q^{11/12} \leqslant T \leqslant q^{3/2}, \\ T^{2/3}q^{1/8+\epsilon} & \text{if } q^{3/2} \leqslant T \leqslant q^{9/4}, \\ T^{1/2}q^{1/2+\epsilon} & \text{if } q^{9/4} \leqslant T. \end{cases}$$

Remark. In [1] under the conditions of Collorally 1 for $q^{2/3} \leqslant T \leqslant q^{83/84}$ it is shown that

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll T^{13/18}q^{5/27+o(1)}.$$

Under this conditions our bound is more precise.

Theorem 2. *Let χ_1 and χ_2 be two primitive Dirichlet characters with prime conductors q_1 and q_2 respectively, $q_1 \leqslant q_2$, $T > 1$ and let $r \geqslant 2$ be an integer. Put*

$$\nu_r := \begin{cases} 1 & \text{if } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$T_r := q_1^{\frac{(r+1)^2}{4r}} q_2^{\frac{r+1}{4r}} (\log q_1)^{r+1} (\log q_2)^{\nu_r r(r+1)+r^2+1}.$$

Then

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll \begin{cases} T^{1-\frac{1}{r}} (q_1 q_2)^{\frac{r+1}{4r^2}} \log^{\frac{1}{r}} q_1 \log^{\frac{1}{r}+\nu_r+1} q_2 & \text{if } (q_1q_2)^{\frac{r+1}{4r}} \leqslant T \leqslant T_r, \\ T^{\frac{r}{r+1}} q_2^{\frac{1}{4r}} (\log q_2)^{\frac{2}{r+1}} & \text{if } T_r \leqslant T, \end{cases}$$

3 Basic notations

Let χ_1 and χ_2 be two primitive Dirichlet characters with conductors q_1 and q_2. Suppose that $q_1 \leqslant q_2$. Set

$$Q := q_1 q_2.$$

For a parameter $T > 0$ we consider a hyperbola

$$\Gamma := \{(x, y) \in \mathbb{R}_+^2 \mid xy = T\}.$$

For a subset $\Omega \subset \mathbb{R}^2$ we define the character sum

$$S(\Omega) := \sum_{(x,y)\in\Omega\cap\mathbb{Z}^2} \chi_1(x)\chi_2(y).$$

For $k \in \mathbb{N}$ we define the value

$$\sigma_k := \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}.$$

In our proofs we will use the following result (see [2]).

Theorem 3 (Burgess)**.** *For any primitive Dirichlet character χ of conductor q and any nonnegative integers M, N we have*

$$\left| \sum_{M<n\leqslant M+N} \chi(n) \right| \leqslant c_\epsilon N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon},$$

where $r \in \{1, 2, 3\}$. If q is a prime number then

$$\left| \sum_{M<n\leqslant M+N} \chi(n) \right| \leqslant c'_\epsilon N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\ln q)^{\frac{1}{r}}$$

for every $r \geqslant 1$.

Suppose I_1 and I_2 are two intervals. Then we define a rectangle $I_1 \times I_2$ as follows:

$$I_1 \times I_2 := \{(x,y) \in \mathbb{R}^2 \mid x \in I_1, y \in I_2\}.$$

We write $|\Pi|$ for area of rectangle Π and $\delta(\Pi) = length(I_1)$ for its width.

Consider rectangles

$$U_0 := (0; \sqrt{T}) \times [0; \sqrt{T}), \tag{3}$$

$$U_k := \left(0; \frac{\sqrt{T}}{2^k}\right) \times [2^{k-1}\sqrt{T}; 2^k\sqrt{T}),$$

where $k = 1, 2, \ldots$ All rectangles U_k have one vertex on Γ.

Suppose that the rectangle

$$\Pi = [x_0, x_1) \times [y_0, y_1)$$

has a vertex (x_1, y_1) on hyperbola Γ, i.e. $x_1 y_1 = T$. Then we define two new rectangles $r(\Pi)$ and $u(\Pi)$ by the following rule:

$$r(\Pi) := \left[x_1, \frac{3x_1 - x_0}{2}\right) \times \left[y_0, \frac{2T}{3x_1 - x_0}\right),$$

$$u(\Pi) := \left[x_0, \frac{x_0 + x_1}{2}\right) \times \left[y_1, \frac{2T}{x_0 + x_1}\right).$$

For $k = 1, 2, 3, \ldots.$ we define rectangles

$$\Pi_k := \left[\frac{\sqrt{T}}{2^k}; \frac{3\sqrt{T}}{2^{k+1}}\right) \times \left[2^{k-1}\sqrt{T}; \frac{2^{k+1}}{3}\sqrt{T}\right) = r(U_k). \tag{4}$$

Define the set $\mathcal{F}_k$ of all rectangles that can be representented in the form $\sigma_1 \cdots \sigma_n \Pi_k$, where $\sigma_i \in \{r, u\}, i = 1, \ldots, n$, for some $n \geqslant 0$.

If rectangle $\Pi \in \mathcal{F}_k$ is represented in the form $\Pi = \sigma_1 \cdots \sigma_l \Pi_k$ then we say that Π is a rectangle of order l.

4 Lemmata

Lemma 1. *Consider rectangle Π. Suppose $P := |\Pi|$ and rectangle's height and width are both greater than 1. Then for every real $\epsilon > 0$ one has*

$$|S(\Pi)| \ll \begin{cases} P^{2/3} Q^{1/9+\epsilon}, \\ P^{1/2} Q^{3/16+\epsilon}. \end{cases}$$

Proof. It is sufficient to apply Burgess' theorem with $r = 3$ in first case and with $r = 2$ in second. $\square$

Lemma 2. *Suppose* $1 \leqslant x_0 < x$, $y := T/x$. *Put*

$$\Delta := x - x_0,$$

$$x_n := x_0 + \Delta\sigma_n = x - \frac{\Delta}{2^n},$$

$n = 1, 2, 3, \dots$ *Consider rectangles of the form*

$$\Phi_n = [x_{n-1}, x_n) \times [y, T/x_n).$$

Then

$$|\Phi_n| = \frac{\Delta^2}{2^{2n}} \frac{y}{x_n}, \quad |u(\Phi_n)| = \frac{|\Phi_n|}{4 \cdot \left(1 - \frac{3}{2} \cdot \frac{\Delta}{x} \cdot \frac{1}{2^n}\right)}.$$

Proof. Let $y_n = \frac{T}{x_n}$ then

$$y_n = \frac{T}{x_0 + \Delta\sigma_n}.$$

The area of rectangle Φ_n is equal to

$$\begin{aligned} |\Phi_n| = (x_n - x_{n-1})(y_n - y) &= \Delta(\sigma_n - \sigma_{n-1})(\frac{T}{x_n} - y) \\ &= \frac{\Delta}{2^n} \frac{T - y(x - \Delta/2^n)}{x_0 + \Delta\sigma_n} = \frac{\Delta^2}{2^{2n}} \frac{y}{x_n}. \end{aligned}$$

So the first equality is proved.

Using the same argument we obtain

$$|u(\Phi_n)| = \frac{\Delta'^2}{4} \frac{y_n}{x_n - \Delta'/2},$$

where $\Delta' = \Delta/2^n$. Therefore

$$|u(\Phi_n)| = \frac{\Delta^2}{4 \cdot 2^{2n}} \cdot \frac{T}{x_n(x_n - \Delta/2^{n+1})}.$$

Hence

$$\frac{|\Phi_n|}{|u(\Phi_n)|} = 4 \cdot \frac{y}{x_n} \cdot \frac{x_n}{T} \cdot (x_n - \Delta/2^{n+1}) = 4 \cdot \frac{1}{x} \left(1 - \frac{\Delta}{2^n} - \frac{\Delta}{2^{n+1}}\right).$$

□

Lemma 3. *Consider rectangle* $\Pi \in \mathcal{F}_k$. *Then*

$$|r(\Pi)| \leqslant |\Pi|/4.$$

Proof. Suppose that under conditions of Lemma 2

$$\Pi = \Phi_1, \quad r(\Pi) = \Phi_2.$$

Then

$$\frac{|\Pi|}{|r(\Pi)|} = 4\frac{x_2}{x_1}.$$

But $x_2 > x_1$ so we obtain Lemma. □

Lemma 4. *Consider rectangle $\Pi \in \mathcal{F}_k$ with vertex (x, y) on the hyperbola Γ, so $xy = T$. Let $\delta = \delta(\Pi)$. Then*

$$|u(\Pi)| \leqslant \frac{|\Pi|}{4\left(1 - \frac{3\delta}{2x}\right)}.$$

Proof. Without loss of generality we can assume that Φ is the first rectangle in the sequence of rectangles from Lemma 2.

Then

$$\frac{\Delta'}{x'} = \frac{2\delta}{x+\delta} = \frac{\delta}{x}\frac{2}{1+\delta/x} \leqslant \frac{2\delta}{x}.$$

Therefore

$$4 \cdot \left(1 - \frac{3}{2} \cdot \frac{\Delta'}{x'} \cdot \frac{1}{2}\right) \geqslant 4 \cdot \left(1 - \frac{3\delta}{2x}\right).$$

$\square$

Lemma 5. *Let $\Pi = \sigma_1 \cdots \sigma_l \Pi_k$. Then*

$$|\Pi| \leqslant \frac{|\Pi_k|}{4^l \prod_{j=1}^{l}(1 - \frac{3}{2}(\frac{2}{3})^l)}.$$

Proof. We will show that the ratio δ/x is reduced by a factor $\geq 3/2$ every time when rectangle Π is replaced by $u(\Pi)$ or $r(\Pi)$.

Case 1. Consider rectangle Π with parameters (δ, x) and rectangle $u(\Pi)$ with parameters (δ', x'). Then $\delta' = \delta/2$ and $x' = x - \delta/2$. Therefore

$$\frac{\delta'}{x'} = \frac{\delta/2}{x - \delta/2} = \frac{\delta}{2x}\frac{1}{1 - \frac{\delta}{2x}} \leqslant \frac{2\delta}{3x},$$

because $\delta/x \leqslant 1/2$ for all rectangles in $\mathcal{F}_k$.

Case 2. Consider rectangle Π with parameters (δ, x) and rectangle $r(\Pi)$ with parameters (δ', x'). Then $\delta' = \delta/2$ and $x' = x + \delta/2$, therefore

$$\frac{\delta'}{x'} = \frac{\delta/2}{x + \delta/2} = \frac{\delta}{2x}\frac{1}{1 + \frac{\delta}{2x}} \leqslant \frac{\delta}{2x} \leqslant \frac{2\delta}{3x}.$$

Lemma is proved by applying Lemma 3 and Lemma 4. $\square$

Lemma 6. *Let real δ, t and T be such that $1/2 < \delta < 1$ and $1 < t < T^\delta$. Set*

$$\Xi_t := \{(x, y) \in \mathbb{R}_+^2 \mid T - 2t \leqslant xy \leqslant T\}.$$

Then $\#(\Xi_t \cap \mathbb{Z}^2) \ll t \ln T$.

Proof. Number of integer points under hyperbola can be estimated by

$$\sum_{x=1}^{T}\left[\frac{T}{x}\right] = T\ln T + (2\gamma - 1)T + O(T^{1/2}).$$

Therefore

$$\#(\Xi_t \cap \mathbb{Z}^2) = T\ln T + (2\gamma-1)T - (T-2t)\ln(T(1-\frac{2t}{T})) - (2\gamma-1)(T-2t) + O(T^{1/2}).$$

Thus, Lemma is proved. $\square$

5 Proof of Theorem 1 for small T

Set $\Omega = \{(x,y) \in \mathbb{R}_+^2 \mid xy < T\}$, and $\Omega_1 = \{(x,y) \in \mathbb{R}_+^2 \mid xy < T, x < \sqrt{T}\}$. Without loss of generality we can estimate only $S(\Omega_1)$.

It is obviously that rectangles from $\mathcal{F}_k, k = 1, 2, \ldots$ together with $U_k, k = 0, 1, 2, \ldots$ cover all the set Ω_1.

Consider $t := T^{3/4} q_2^{1/12}$ and real $\eta > 0$. Set

$$W_t := \{(x,y) \in \mathbb{R}_+^2 \mid xy < T,\ y \leqslant t,\ x \leqslant \sqrt{T}\}, \tag{5}$$

$$W_t' := \{(x,y) \in \mathbb{R}_+^2 \mid xy < T,\ y \geqslant t\}, \tag{6}$$

$$\Xi_t = \{(x,y) \in \mathbb{R}_+^2 \mid T - 2t \leqslant xy \leqslant T\}. \tag{7}$$

The number of integer points in Ξ_t is bounded by $\#(\Xi \cap \mathbb{Z}^2) \ll tQ^\eta$.

Consider a rectangle $\Pi \in \mathcal{F}_k$ such that $\Pi \subset W$ and let (x_0, y_0) be its left bottom vertex. Set $\delta = \delta(\Pi)$.

Let

$$2\delta = \frac{T}{y_0} - x_0 \leqslant 2.$$

Then $\Pi \subset \Xi_t$. Indeed

$$T - x_0 y_0 \leqslant 2y_0 \leqslant 2t,$$

therefore $T - 2t \leqslant x_0 y_0$.

Now we estimate $S(W_t)$.

For rectangles $\Pi \in \mathcal{F}_k$, $\Pi \subset W_t$ with $\delta(\Pi) \geqslant 1$ we apply Lemma 1. All other rectangles are lying in Ξ_t.

The sum $S(\Xi_t)$ is trivially bounded by the number of integer points in Ξ_t.

The number of rectangles Π of order l is equal to 2^l. The area of such a rectangle Π is bounded by $|\Pi| \ll |\Pi_k|/4^l$.

Thus, we have the following bound for the character sum over all rectangles Π of order l and with $\delta(\Pi) \geqslant 1$:

$$S(\Pi^l) \ll T^{2/3} Q^{1/9+\eta} 4^{-2l/3} 2^l.$$

As the sum $\sum_{l=0}^{\infty} 4^{-2l/3} 2^l$ converges we see that the character sum over all rectangles Π with $\delta(\Pi) \geqslant 1$ is bounded by $\ll T^{2/3} Q^{1/9+\eta}$.

Therefore

$$S(W_t) \ll \max(T^{2/3} Q^{1/9+\eta}, tQ^\eta). \tag{8}$$

In order to estimate $S(W_t')$ we use Burgess' Lemma with $r = 3$:

$$S(W_t') \ll \sum_{x=1}^{T/t} (T/x)^{2/3} q_2^{1/9+\eta} \ll T^{2/3} (T/t)^{1/3} q_2^{1/9+\eta},$$

therefore

$$S(W_t') \ll T t^{-1/3} q_2^{1/9+\eta}. \tag{9}$$

So

$$S(\Omega_1) \ll \max(T^{2/3} Q^{1/9+\eta}, tQ^\eta, T t^{-1/3} q_2^{1/9+\eta}).$$

Using the definition of parameter t we obtain the following result. If $T \leqslant q_1^{4/3}q_2^{1/3}$ then $S(\Omega) \ll T^{2/3}Q^{1/9+\eta}$. If $T \geqslant q_1^{4/3}q_2^{1/3}$ then $S(\Omega) \ll tQ^\eta = T^{3/4}q_2^{1/12+\eta}$.

6 Proof of Theorem 1 for large T

Set
$$t := T^{2/3}q_2^{1/8}. \tag{10}$$

As before, we use the sets W_t, W'_t and Ξ_t defined in (5), (6) and (7).

The only difference between this case and previous one is the convergence argument. The sum over all rectangles of order l can be estimated by $T^{1/2}Q^{3/16+\eta/2}$. Therefore the sum
$$\sum_{l=0}^{\infty} T^{1/2}Q^{3/16+\eta/2}$$
does not converge. But it is easy to see that if $l \gg \log T$ then every rectangle Π of order l lies in the set Ξ_t. So we have
$$S(W_t) \ll \max(T^{1/2}Q^{3/16+\eta}, tQ^\eta). \tag{11}$$

Applying Burgess' Lemma with $r = 2$ we obtain
$$S(W'_t) \ll \sum_{x=1}^{T/t} (T/x)^{1/2}q_2^{3/16+\eta} \ll T^{1/2}(T/t)^{1/2}q_2^{3/16+\eta},$$
therefore
$$S(W'_t) \ll Tt^{-1/2}q_2^{3/16+\eta}. \tag{12}$$

Inserting (10) into (11) and (12), we have if $T \leqslant q_1^{9/8}q_2^{3/8}$ then $S(\Omega) \ll T^{1/2}Q^{3/16+\eta}$, and if $T \geqslant q_1^{9/8}q_2^{3/8}$ then $S(\Omega) \ll tQ^\eta = T^{2/3}q_2^{1/8+\eta}$.

7 Prime moduli

Set $r \geqslant 2$ and $t := T^{\frac{r}{r+1}} q_2^{\frac{1}{4r}} \log^{\frac{1-r}{r+1}} q_2$. We use sets defined by (5), (6) and (7) again.

Our argument to estimate $S(W_t)$ is similar. For $k \geqslant 1$ there exist 2^l rectangles of order l. Applying Lemma 5, we have that the character sum over all rectangles of order l is bounded by
$$S(\Pi^l) \ll T^{1-\frac{1}{r}} Q^{\frac{r+1}{4r^2}} (\log q_1)^{\frac{1}{r}} (\log q_2)^{\frac{1}{r}} 4^{-(1-\frac{1}{r})l} 2^l.$$

We consider two cases.

Case 1 ($r \geqslant 3$). The sum $\sum_{l=0}^{\infty} 4^{-(1-\frac{1}{r})l}2^l$ converges, so
$$\sum_{l=0}^{\infty} S(\Pi^l) \ll T^{1-\frac{1}{r}} Q^{\frac{r+1}{4r^2}} (\log q_1)^{\frac{1}{r}} (\log q_2)^{\frac{1}{r}}.$$

Case 2 ($r = 2$). The sum $\sum_{l=0}^{\infty} 4^{-(1-\frac{1}{r})l} 2^l$ does not converge. In this case it is sufficient to take only first $\ll \log(T) \ll \log(q_2)$ values of l.

There are only $\ll \log T \ll \log q_2$ rectangles from $\mathcal{F}_k$ lying lower the line $y = t$. So

$$S(W_t') \ll \max(T^{1-\frac{1}{r}} Q^{\frac{r+1}{4r^2}} \log^{1/r} q_1 \log^{\frac{1}{r}+\nu_r+1} q_2, t \log q_2).$$

By Burgess' Lemma, we have

$$S(W_t') \ll \sum_{x=1}^{T/t} (T/x)^{1-\frac{1}{r}} q_2^{\frac{r+1}{4r^2}} (\log q_2)^{\frac{1}{r}} \ll T t^{-\frac{1}{r}} q_2^{\frac{r+1}{4r^2}} (\log q_2)^{\frac{1}{r}}.$$

8 Proof of Collorally 1

First three inequalities are immediate consequences of Theorem 1.

We obtain the last inequality by applying Burgess' Lemma with $r = 1$. We begin with splitting the sum over points under hyperbola into three parts:

$$\Omega_1 = \{(x, y) \in \mathbb{R}_+^2 \mid xy < T, x < \sqrt{T}\},$$

$$\Omega_2 = \{(x, y) \in \mathbb{R}_+^2 \mid xy < T, y < \sqrt{T}\},$$

and U_0, defined by (3).

Applying Burgess' Lemma with $r = 1$, we have

$$S(\Omega_1) \ll \sqrt{T} q^{1/2+\epsilon},$$

and

$$S(U_0) \ll q^{1+\epsilon}.$$

We are interested in the case $T \geqslant q^{3/2}$. So $\sqrt{T} q^{1/2+\epsilon} \geqslant q^{1+\epsilon}$ and we obtain the collorally.

References

[1] William D. Banks, Igor E. Shparlinski, *Sums with convolutions of Dirichlet characters,* Manuscripta Math. 133, 105-114 (2010)

[2] Iwaniec, H., Kowalski, E. *Analytic Number Theory.* American Mathematical Society, Providence (2004)